On the stability of stochastic dynamic systems and their use in econometrics

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2019

document version
Publisher's PDF, also known as Version of record

Chapter 2

Stationarity, Ergodicity and Mixing of Resetting Time Series

2.1 Introduction

Since the popularisation of linear time series models such as the autoregressive moving average model (Box et al., 1970) for level modelling and the autoregressive conditional heteroskedasticity (Engle, 1982) model for volatility modelling much innovation has been made. Present-day, fitting time series with nonlinear models has become increasingly common. A selection of such nonlinear methods that can be applied to both level and volatility modelling are regime switching models (Hamilton, 1989), threshold models (Tong and Lim, 1980; Zakoian, 1994) and score driven models (Creal et al., 2013; Harvey, 2013).

Stability properties of both data generated by a model and unobserved parameters when filtering data are very useful. Knowing when a time series is stationary ergodic with mixing properties allows one to apply limit theorems to obtain consistency and asymptotic normality of estimators. However, ensuring stability of econometric models becomes increasingly harder as the models get more complicated. Nonlinear dynamics imply that the theory on Lyapunov exponents as developed in Bougerol and Picard (1992a,b) cannot be used. Therefore one has to resort to more involved methods such as Markov chain theory and geometric ergodicity (Meyn and Tweedie, 2012) or stochastic recurrence equation theory as developed in Bougerol (1993) and Straumann (2005). These methods ensure stability by imposing that the updating function satisfies certain contraction or bounded growth or drift properties. See for example Cline and Pu (1999) and Saïdi and Zakoian (2006) or Blasques et al. (2014) and Straumann (2005) for the application of these conditions to various models.

This paper derives general stability conditions for a large collection of models that are typically either infeasible or less efficient (in the sense of a smaller parameter space) to analyse with the existing methods. This collection of models is characterised by a property that we denote resetting, which requires that the model has a positive probability to update to a fixed, but possibly stochastic, state, irrespective of its past values. The resetting condition allows for very wild sample path behaviour between resetting times, as the reset ensures that the sample path will return to a stable base line. That means that we can include typically unstable dynamics such as explosive or very discontinuous updates in the time series. The framework lends itself naturally to regime switching models, where we are then free to make all but one regime as unstable as we want as long as we ensure that the last regime enforces a reset.

The resetting condition might appear to be restrictive at first, but is often satisfied in time series where sudden drops or increases are observed. Typical examples of such time series are stocks exhibiting financial bubbles, where the crash of the bubble is the moment where the time series resets. See for example the model in Blasques et al. (2018b) and Chapter 3 of this thesis that is developed to describe the Bitcoin/USD exchange rate studied in Hencic and Gouriéroux (2015). There the exchange rate X_t is modelled as the sum of a stationary ergodic process μ_t and a nonnegative bubble process b_t, where

$$b_t = (\omega + \alpha b_{t-1})\mathbb{1}\{b_{t-1} < k(\mu_t - c)\}$$

with $\omega, \alpha > 0$ and $k, c \in \mathbb{R}$. This model consists out of two regimes: one autoregressive regime $b_t = \omega + \alpha b_{t-1}$ and one collapsing regime $b_t = 0$. The bubble process b_{t-1} is nonnegative, so if the innovation $\mu_t < c$, then the indicator function does not hold for any possible value of b_{t-1} and hence the bubble process will collapse/reset regardless of its past values. Note that the stability conditions allow the autoregressive parameter α to be greater than one in this model, in fact this is encouraged to describe bubble behaviour.

This is something that is normally associated with unstable behaviour in autoregressive processes.

An example of collapsing, and thus resetting, volatility dynamics can be found in a model used by Saïdi and Zakoian (2006) to study the real financial time series discussed in Saïdi (2003). They define the dynamics of a heteroskedastic time series $(\epsilon_t)_{t\in\mathbb{Z}}$ as

$$\begin{aligned} \epsilon_t &= \sigma_t \eta_t, \\ \sigma_t^2 &= \omega + \alpha \epsilon_{t-1}^2 \mathbb{1}\left\{\epsilon_{t-1}^2 > k\epsilon_{t-2}^2\right\}, \end{aligned} \tag{2.1}$$

where $(\eta_t)_{t\in\mathbb{Z}}$ is a strictly stationary and ergodic sequence of random variables, the parameters α and k are nonnegative and ω is positive. Similarly to the previous example the parameter α is allowed to be greater than one and model (2.1) consists of two regimes. One regime is the traditional ARCH(1) update $\sigma_t^2 = \omega + \alpha\epsilon_{t-1}^2$ and the other is the conditional homoskedastic model $\sigma_t^2 = \omega$. The model changes from the constant volatility regime to the ARCH(1) specification when the relative variation $\epsilon_{t-1}^2/\epsilon_{t-2}^2$ becomes large, indicating a setting in which it is more likely for the volatility to be time varying. The collapse condition is harder to discern in this model, but occurs when two consecutive η's are much smaller than their predecessors. Saïdi and Zakoian (2006) analyse the stability of their model (2.1) using Markov chain theory. They show the existence of a stationary and β-mixing solution, under the condition that the distribution of the underlying process $(\eta_t)_{t\in\mathbb{Z}}$ is independent and identically distributed (iid), has strict positive density and fixed moments $\mathbb{E}(\eta_t) = 0$ and $\mathbb{E}(\eta_t^2) = 1$. Using our approach we can show the existence of a unique, stationary and ergodic or φ-mixing solution, to which any sample path converges. The assumptions needed to get these results are less strict than those imposed in Saïdi and Zakoian (2006). Our method also allows for extensions of the model with minimum additional theoretical work.

The rest of the paper is structured as follows. Section 2.2 discusses stability conditions for random functions on separable Banach spaces and states our results in their most general form. Section 2.3 illustrates how to apply the theory to a practical model by considering a generalisation of model (2.1) and deriving the stability conditions for various distributional assumptions. Moreover, we showcase the ease of application by deriving the conditions for various practical examples including leverage effects and robust news impact curves and also show how the method can be used to derive moment bounds.

2.2 Stability results

In this section we prove our main results for resetting time series. Theorem 2.2.1 shows the existence of a stationary ergodic solution to which all sample paths converge and Theorem 2.2.4 discusses the existence of a solution that is φ-mixing at geometric rate. We base our treatment of stability on *stochastic recurrence equations* (SREs) as is done in Straumann (2005) and Straumann and Mikosch (2006). The main advantage of stochastic recurrence equation (SRE) techniques is that they are very general. For example, proposition 7.6 in Kallenberg (2002) proves that any homogeneous Markov chain can be seen as a solution to a SRE. We refer the reader to Diaconis and Freedman (1999) for a thorough overview of SREs.

We will work with random elements on Banach spaces. This allows us to describe time varying variables in econometric models as functions of the model parameters, which can be used to obtain stronger inference results as is done in Straumann and Mikosch (2006). Let S be a closed subset of a separable Banach space equipped with a norm $\|\cdot\|$ and Borel sigma-algebra $\mathcal{B}(S)$ and let $(E, \mathcal{E})$ be a measurable space. Let $(\eta_t)_{t\in\mathbb{Z}}$ be a sequence of stochastic elements taking values in E and let $\phi : S \times E \to S$ be a measurable map. Then we can define a sequence of random functions $(\phi_t)_{t\in\mathbb{Z}}$ by setting $\phi_t := \phi(\cdot, \eta_t)$. Let T be either $\mathbb{Z}$ or $\mathbb{N}$. A stochastic process $(X_t)_{t\in T}$ taking values in S that satisfies

$$X_{t+1} = \phi_t(X_t) \qquad \forall t \in T \tag{2.2}$$

is said to be a *solution* to the SRE associated with $(\phi_t)_{t\in\mathbb{Z}}$ if $T = \mathbb{Z}$, and a *partial solution* if $T = \mathbb{N}$. We now construct a specific possible solution $(Y_t)_{t\in\mathbb{Z}}$ to (2.2) by using the backward iterates defined as $\phi_t^{(0)} = \mathrm{Id}_S$ and

$$\phi_t^{(m)} = \phi_t \circ \phi_{t-1} \cdots \circ \phi_{t-m+1}, \qquad m \in \mathbb{N}.$$

Let $x \in S$ be an element such that

$$Y_{t+1} := \lim_{m\to\infty} \phi_t^{(m)}(x) \tag{2.3}$$

exists almost surely for all $t \in \mathbb{Z}$. Bougerol (1993) and Straumann and Mikosch (2006) show that this is the case under appropriate regularity conditions involving the contracting behavior of each ϕ_t and the distribution of $(\phi_t)_{t\in\mathbb{Z}}$. Moreover, they show that the sequence of limits $(Y_t)_{t\in\mathbb{Z}}$ is then the unique ergodic solution to (2.2) and that any partial solution converges to this unique one at a geometric rate as $t \to \infty$. In this article we pursue a similar approach, we also focus on the limit of the backward iterates in (2.3), show that it is well defined and that the resulting sequence $(Y_t)_{t\in\mathbb{Z}}$ possesses the right properties. However, we rely on considerably different conditions and replace the contraction condition in Bougerol (1993) with a new resetting condition.

Assumption A. The sequence $(\phi_t)_{t\in\mathbb{Z}}$ satisfies the following conditions:

A1. The function ϕ is $\mathcal{B}(S) \times \mathcal{E}/\mathcal{B}(S)$ measurable.

A2. The sequence $(\eta_t)_{t\in\mathbb{Z}}$ is strictly stationary ergodic.

A3. There exists an $M \in \mathbb{N}$ and an event $A \in \mathcal{E}^M$ such that $(\eta_t, \eta_{t-1}, \dots, \eta_{t-M+1}) \in A$ with positive probability and

$$(\eta_t, \eta_{t-1}, \dots, \eta_{t-M+1}) \in A \quad \Rightarrow \quad \phi_t^{(M)}(x) = \phi_t^{(M)}(y) \quad \forall x, y \in S.$$

Condition A1 is rather weak and designed to ensure that backward iterates of $(\phi_t)_{t\in\mathbb{Z}}$ evaluated at any point $x \in S$ are proper random variables in S. Condition A2 is common in the literature on SREs, note that it is less strict than assuming that the sequence $(\eta_t)_{t\in\mathbb{Z}}$ is independent and identically distributed. An in depth discussion on stationarity and ergodicity can, for example, be found in chapter one of Krengel (1985). Condition A3 is the resetting condition and states that there exists an event over M periods that guarantees that the corresponding M'th iterate is constant over S, but not necessarily over E^M. This implies that $\phi_t^{(M)}$ is constant for a given realisation of $(\eta_t)_{t\in\mathbb{Z}}$ in A and thus resets to one constant value, irrespective of its argument and hence the past values of a solution to the SRE.

Under Assumption A we can prove that the limit of the backward iterates (2.3) exists, by showing that the sequence of backward iterates $(\phi_t^{(m)}(x))_{m\in\mathbb{N}}$ is almost surely eventually constant. The proof relies on the fact that events of positive probability occur infinitely often over time in a strictly stationary ergodic sequence. Therefore the event $(\eta_t, \eta_{t-1}, \ldots, \eta_{t-M+1}) \in A$ occurs for infinitely many $t \in \mathbb{Z}$ and thus the limit of the backward iterates trivially exists. Uniqueness and convergence of paths follow from the same observation, since any two paths in model (2.2) will coincide at all such t, and therefore must be the same (eventually).

Theorem 2.2.1. *Let Assumption A hold and $x \in S$. Then the sequence $(\phi_t^{(m)}(x))_{m\in\mathbb{N}}$ is almost surely eventually constant for all $t \in \mathbb{Z}$. Consequently, $(Y_t)_{t\in\mathbb{Z}}$ is well defined, strictly stationary ergodic and the unique solution to* (2.2). *Moreover, for any partial solution $(\tilde{Y}_t)_{t\in\mathbb{N}}$ and function $f : \mathbb{N} \to \mathbb{R}$ we have $\lim_{t\to\infty} f(t)\|Y_t - \tilde{Y}_t\| = 0$.*

We have to discuss some preliminary results on strictly stationary ergodic (SE) sequences before we can prove Theorem 2.2.1. One reason that SE sequences play a big role in time series analysis is that they satisfy the conditions needed for Birkhoff's ergodic theorem, Birkhoff (1931). This theorem applied to an SE sequence of real valued random variables $(X_t)_{t\in\mathbb{N}}$ states that if $\mathbb{E}|X_1| < \infty$, then almost surely

$$\lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} X_t = E(X_1).$$

SE sequences are also easy to manipulate to create new SE sequences. We provide two results from Straumann (2005).

Lemma 2.2.2. *Let $(E, \mathcal{E})$ and $(\tilde{E}, \tilde{\mathcal{E}})$ be two measurable spaces, let $(X_t)_{t\in\mathbb{Z}}$ be an SE sequence of E-valued random elements and let $f : E^{\mathbb{N}} \to \tilde{E}$ be a $\mathcal{E}^{\mathbb{N}}/\tilde{\mathcal{E}}$ measurable function. Then the sequence of $\tilde{E}$-valued random elements $(\tilde{X}_t)_{t\in\mathbb{Z}}$ defined as $\tilde{X}_t = f(X_t, X_{t-1}, \ldots)$ is SE.*

PROOF. See proposition 2.1.1 in Straumann (2005). ∎

Lemma 2.2.3. *Let $(E, \mathcal{E})$ be a measurable space and let $(S, \mathcal{B}(S))$ be a closed subset of a separable Banach space endowed with its Borel sigma-algebra. Let $(X_t)_{t\in\mathbb{Z}}$ be a SE sequence of E-valued random elements and let $(f_m)_{m\in\mathbb{N}}$ be a sequence of functions $E^{\mathbb{N}} \to S$ that are $\mathcal{E}^{\mathbb{N}}/\mathcal{B}(S)$ measurable. Suppose that there exists a $t \in \mathbb{Z}$ such that*

$$\lim_{m\to\infty} f_m(X_t, X_{t-1}, \ldots)$$

exists almost surely. Then there exists a function $f : E^{\mathbb{N}} \to S$ that is $\mathcal{E}^{\mathbb{N}}/\mathcal{B}(S)$ measurable and satisfies

$$\tilde{X}_t := \lim_{m\to\infty} f_m(X_t, X_{t-1}, \ldots) = f(X_t, X_{t-1}, \ldots)$$

for all $t \in \mathbb{Z}$. Moreover, the sequence of S-valued random elements $(\tilde{X}_t)_{t\in\mathbb{Z}}$ is SE.

PROOF. See corollary 2.1.3 in Straumann (2005). ■

PROOF OF THEOREM 2.2.1. Fix a $t \in \mathbb{Z}$. We begin by proving that $(\phi_t^{(m)}(x))_{m\in\mathbb{N}}$ is almost surely eventually constant. Define for $s \geq 0$,

$$I_s = \mathbb{1}\{(\eta_{t-s}, \eta_{t-s-1}, \ldots, \eta_{t-s-M+1}) \in A\}.$$

The sequence $(I_s)_{s\geq 0}$ is SE by Lemma 2.2.2. Then, by Birkhoff's ergodic theorem, almost surely

$$\lim_{n\to\infty} \frac{1}{n} \sum_{s=0}^{n-1} I_s = E(I_0) = \mathbb{P}((\eta_t, \eta_{t-1}, \ldots, \eta_{t-M+1}) \in A) > 0.$$

This implies that the event $I_s = 1$ occurs almost surely for infinitely many $s \geq 0$. Therefore we can choose the smallest such s, note that it is a random variable, and conclude that

$$\phi_t^{(m)}(x) = \phi_t^{(s)}\left(\phi_{t-s}^{(m-s)}(x)\right) = \phi_t^{(s)}\left(\phi_{t-s}^{(M)}\left(\phi_{t-s-M}^{(m-s-M)}(x)\right)\right) = \phi_t^{(s)}\left(\phi_{t-s}^{(M)}(x)\right)$$

for all $m \geq s + M$. It follows by Lemma 2.2.3 that the sequence $(Y_t)_{t\in\mathbb{Z}}$ is well defined and SE. Moreover, for $s = 0$ we get $Y_{t+1} = \phi_t^{(M)}(x) = \phi_t^{(M)}(Y_{t-M+1}) = \phi_t(Y_t)$, while for $s \geq 1$ we have

$$\begin{aligned} Y_{t+1} &= \lim_{m\to\infty} \phi_t^{(m)}(x) = \phi_t^{(s)}\left(\phi_{t-s}^{(M)}(x)\right) = \phi_t\left(\phi_{t-1}^{(s-1)}\left(\phi_{t-s}^{(M)}(x)\right)\right) \\ &= \phi_t\left(\lim_{m\to\infty} \phi_{t-1}^{(m)}(x)\right) = \phi_t(Y_t). \end{aligned}$$

Therefore $(Y_t)_{t\in\mathbb{Z}}$ is a solution to (2.2). If $(X_t)_{t\in\mathbb{Z}}$ is any other solution to (2.2), then

$$X_{t+1} = \phi_t^{(s)}\left(\phi_{t-s}^{(M)}(X_{t-s-M+1})\right) = \phi_t^{(s)}\left(\phi_{t-s}^{(M)}(Y_{t-s-M+1})\right) = Y_{t+1},$$

and hence it is identical to $(Y_t)_{t\in\mathbb{Z}}$.

It remains to prove the final statement. Similarly as before, we can almost surely find an $s > M-1$ such that $(\eta_s, \eta_{s-1}, \ldots, \eta_{s-M+1}) \in A$ and thus

$$Y_{t+1} = \phi_t^{(t-s)}\left(\phi_s^{(M)}(Y_{t-s-M+1})\right) = \phi_t^{(t-s)}\left(\phi_s^{(M)}\left(\tilde{Y}_{t-s-M+1}\right)\right) = \tilde{Y}_{t+1}$$

for all $t \geq s$. We conclude that

$$\lim_{t\to\infty} f(t)\|Y_t - \tilde{Y}_t\| = 0,$$

irrespective of the function f, because $\|Y_t - \tilde{Y}_t\|$ is almost surely eventually zero. ■

A consequence of Theorem 2.2.1 is that we can derive sufficient conditions for the process $(Y_t)_{t\in\mathbb{Z}}$ to be φ-mixing. Let $(X_t)_{t\in\mathbb{Z}}$ be a stationary process and let $\mathcal{F}_s^t$, for $-\infty \leq s < t \leq \infty$, denote the sigma algebra generated by $(X_s, X_{s+1}, \ldots, X_t)$. Then the φ-mixing coefficients for $(X_t)_{t\in\mathbb{Z}}$ are given by

$$\varphi_X(t) = \sup_{C\in\mathcal{F}_{-\infty}^0,\, D\in\mathcal{F}_t^\infty,\, \mathbb{P}(C)>0} |\mathbb{P}(D|C) - \mathbb{P}(D)|$$

and the process is called φ-mixing if $\varphi_X(t) \to 0$ as $t \to \infty$.

Theorem 2.2.4. *Suppose Assumption A holds and that additionally $(\eta_t)_{t\in\mathbb{Z}}$ is φ-mixing with geometric rate. Then $(Y_t, \eta_t)_{t\in\mathbb{Z}}$ is φ-mixing with geometric rate.*

The proof will depend on Theorem 2.2.1 as follows. Usually Y_{t+1} depends on the entire past $(\eta_t, \eta_{t-1}, \ldots)$. However, if the event $(\eta_{t-s}, \eta_{t-s-1}, \ldots, \eta_{t-s-M+1}) \in A$ occurs for some $s \geq 0$, then $Y_{t+1} = \phi_t^{(t-s)}\left(\phi_s^{(M)}(x)\right)$ and thus Y_{t+1} depends only on $(\eta_t, \ldots, \eta_{t-s-M+1})$. Therefore it will be enough to show that the probability that s is large vanishes at a geometric rate. To show this we need the following two lemma's.

Lemma 2.2.5. *Let $(E, \mathcal{E})$ and $(\tilde{E}, \tilde{\mathcal{E}})$ be two measurable spaces, let $(X_t)_{t\in\mathbb{Z}}$ be a sequence of E-valued random elements that is φ-mixing (with geometric rate). For a $m \in \mathbb{N}$ we denote $f : E^m \to \tilde{E}$ to be a $\mathcal{E}^m/\tilde{\mathcal{E}}$ measurable function. Then the sequence of $\tilde{E}$-valued random elements $(\tilde{X}_t)_{t\in\mathbb{Z}}$ defined as*

$$\tilde{X}_t = f(X_t, \ldots, X_{t-m})$$

is φ-mixing (with geometric rate).

PROOF. The sigma-algebra generated by $(\ldots, \tilde{X}_{-1}, \tilde{X}_0)$ is contained in the sigma-algebra generated by $(\ldots, X_{-1}, X_0)$. Similarly, the sigma-algebra generated by $(\tilde{X}_t, \tilde{X}_{t+1}, \ldots)$ is contained in the sigma-algebra generated by $(X_{t-m}, X_{t-m+1}, \ldots)$. Therefore $\varphi_{\tilde{X}}(t) \leq \varphi_X(t-m)$ for all $t \geq m$. ■

Lemma 2.2.6. *Let $(E, \mathcal{E})$ be a measurable space and let $(X_i)_{i\in\mathbb{Z}}$ be a strictly stationary sequence of E-valued random elements that is φ-mixing. Then for any $B \in \mathcal{E}$ such that $\mathbb{P}(X_1 \notin B) < 1$, we have*

$$\mathbb{P}\left(\bigcap_{i=1}^{t}\{X_i \notin B\}\right) \to 0 \qquad as \qquad t \to \infty$$

at a geometric rate.

PROOF. For a real number $z \in \mathbb{R}$ we write $\lfloor z \rfloor$ to denote the largest integer that is not larger than z. Also, we use the ; symbol to denote joint probabilities. For any integer

$k \leq t$ we have

$$
\begin{aligned}
&\mathbb{P}(X_t \notin B; \ldots; X_1 \notin B) \\
&\quad = \prod_{i=0}^{\lfloor t/k \rfloor - 1} \mathbb{P}(X_{t-ik} \notin B; \ldots; X_{t-(i+1)k+1} \notin B \mid X_{t-(i+1)k} \notin B; \ldots; X_1 \notin B) \\
&\quad \leq \prod_{i=0}^{\lfloor t/k \rfloor - 1} \mathbb{P}(X_{t-ik} \notin B \mid X_{t-(i+1)k} \notin B; \ldots; X_1 \notin B) \\
&\quad \leq \prod_{i=0}^{\lfloor t/k \rfloor - 1} \mathbb{P}(X_{t-ik} \notin B) + \varphi_X(k) \\
&\quad = (\mathbb{P}(X_1 \notin B) + \varphi_X(k))^{\lfloor t/k \rfloor - 1}.
\end{aligned}
$$

Choose k big enough such that $\mathbb{P}(X_1 \notin B) + \varphi_X(k) < 1$, which can be done since $\varphi_X(k) \to 0$ as $k \to \infty$. Note that if any of the events that we conditioned on has probability zero, then the lemma follows immediately. ■

PROOF OF THEOREM 2.2.4. For $-\infty \leq s < t \leq \infty$ we write $\mathcal{G}_s^t$ to denote the sigma algebra generated by $(\eta_s, \eta_{s+1}, \ldots, \eta_t)$ and $\mathcal{H}_s^t$ to denote the sigma algebra generated by $((Y_s, \eta_s), (Y_{s+1}, \eta_{s+1}), \ldots, (Y_t, \eta_t))$. Fix $t \in \mathbb{N}$ and let $s \geq 0$ again be the random variable that denotes the smallest number such that $(\eta_{t-s}, \eta_{t-s-1}, \ldots, \eta_{t-s-M+1}) \in A$ occurs. Then we have $Y_{t+1} = \lim_{m\to\infty} \phi_t^{(m)}(x) = \phi_t^{(t-s)}\left(\phi_s^{(M)}(x)\right)$. Therefore, for a $B \in \mathcal{B}(S)$ and a $k \geq 0$ the event $\{Y_{t+1} \in B\ ;\ s \leq k\} \in \mathcal{G}_{t-k-M+1}^t$, since $\{s \leq k\} \in \mathcal{G}_{t-k-M+1}^t$. Similarly, for any $D \in \mathcal{H}_t^\infty$ the event $D \cap \{s \leq t/2 - M + 1\} \in \mathcal{G}_{\lceil t/2 \rceil}^\infty$, where we write $\lceil z \rceil$ to denote the smallest integer that is not smaller than z. It follows for $C \in \mathcal{H}_{-\infty}^0 \subseteq \mathcal{G}_{-\infty}^0$, by partitioning on $s \leq t/2 - M + 1$ and its complement, that

$$|\mathbb{P}(D|C) - \mathbb{P}(D)| \leq \varphi_\eta(\lceil t/2 \rceil) + \mathbb{P}(D\ ;\ s > t/2 - M + 1|C) + \mathbb{P}(D\ ;\ s > t/2 - M + 1).$$

Since $\{s > t/2 - M + 1\} \in \mathcal{G}_{\lceil t/2 \rceil}^t$ we get

$$\mathbb{P}(D\ ;\ s > t/2 - M + 1|C) \leq \mathbb{P}(s > t/2 - M + 1|C) \leq \mathbb{P}(s > t/2 - M + 1) + \varphi_\eta(\lceil t/2 \rceil).$$

It follows that

$$\varphi_{(Y,\eta)}(t) \leq 2\varphi_\eta(\lceil t/2 \rceil) + 2\mathbb{P}(s > t/2 - M + 1).$$

The first term goes geometrically fast to zero by assumption. For the second part we define $X_t = (\eta_t, \eta_{t-1}, \ldots, \eta_{t-M+1})$. Then $(X_t)_{t\in\mathbb{Z}}$ is φ-mixing by Lemma 2.2.5. Therefore, by Lemma 2.2.6, and the fact that $\mathbb{P}(X_t \in A) > 0$, we have

$$\mathbb{P}(s > t/2 - M + 1) = \mathbb{P}\left(\bigcap_{i=0}^{\lceil t/2 \rceil} \{X_{t-i} \notin A\} \right) \to 0$$

geometrically fast as $t \to \infty$. ■

2.3 Application to heteroscedastic volatility modelling.

We now introduce a general nonlinear ARCH model that contains the model of Saïdi and Zakoian (2006) and illustrate how to apply our main results of Section 2.2. Let $u : \mathbb{R}^2 \to [0, \infty)$ be a nonnegative Borel measurable function that possibly depends on a vector of parameters θ that lie in a parameter space Θ. The general model of interest is given by

$$\begin{aligned} \epsilon_t &= \sigma_t \eta_t, \\ \sigma_t^2 &= \omega + u(\epsilon_{t-1}, \sigma_{t-1}^2; \theta) \mathbb{1}\left\{\epsilon_{t-1}^2 > k\epsilon_{t-2}^2\right\}, \end{aligned} \tag{2.4}$$

where ω and k are strictly positive. The generalisation compared to (2.1) is that we replace the term $\alpha\epsilon_{t-1}^2$ with a general updating function u. We discuss model (2.1) and other examples in Section 2.3.1.

We start by analysing the dynamics concerning the time varying volatility. Given that u is nonnegative we immediately see that any possible solution to (2.4) must satisfy $\sigma_t^2 \in I := [\omega, \infty)$. Assuming that the model is well specified, we get

$$\sigma_t^2 = \omega + \tilde{u}(\sigma_{t-1}^2, \eta_{t-1}; \theta) \mathbb{1}\left\{\sigma_{t-1}^2\eta_{t-1}^2 > k\sigma_{t-2}^2\eta_{t-2}^2\right\}, \tag{2.5}$$

where $\tilde{u}(\sigma_{t-1}^2, \eta_{t-1}; \theta) = u(\epsilon_{t-1}(\sigma_{t-1}, \eta_{t-1}), \sigma_{t-1}^2; \theta)$. Our analysis will focus on this model, since any solution to (2.5) can be used to create a solution to (2.4). Note that σ_t^2 depends both on σ_{t-1}^2 and σ_{t-2}^2. The random functions $(\phi_t)_{t\in\mathbb{Z}}$ associated with (2.5) will therefore be defined on I^2 and are given by

$$\phi_{t-1}(x, y) = \phi(x, y, \eta_{t-1}, \eta_{t-2}) = \omega + \tilde{u}(x, \eta_{t-1}; \theta)\mathbb{1}\left\{x\eta_{t-1}^2 > ky\eta_{t-2}^2\right\}.$$

Unfortunately these are not in the framework of the SRE theory in Section 2.2, since $\phi : I^2 \times \mathbb{R}^2 \to I$. Therefore we will look at the two dimensional model

$$(\sigma_t^2, \sigma_{t-1}^2) = (\phi_{t-1}(\sigma_{t-1}^2, \sigma_{t-2}^2), \sigma_{t-1}^2), \tag{2.6}$$

which has state space $S := I^2$. The random functions associated with (2.6) are given by

$$\psi_{t-1}(x, y) = \psi(x, y, \eta_{t-1}, \eta_{t-2}) = (\phi_{t-1}(x, y), x).$$

Define $\phi_t^{(-1)}(x, y) = y$ and $\phi_t^{(0)}(x) = x$, then the backward iterates for $m \in \mathbb{N}$ are given by

$$\begin{aligned}
\phi_t^{(m)}(x, y) &= \phi_t\left(\phi_{t-1}^{(m-1)}(x, y), \phi_{t-2}^{(m-2)}(x, y)\right), \\
\psi_t^{(m)}(x, y) &= \left(\phi_t^{(m)}(x, y), \phi_{t-1}^{(m-1)}(x, y)\right).
\end{aligned}$$

We now state the weakest assumption for our nonlinear ARCH model that ensures we satisfy Assumption A and therefore obtain the results from Theorems 2.2.1 and 2.2.4. This result is derived in Theorem 2.3.2.

Assumption B.

B1. The sequence $(\eta_t)_{t\in\mathbb{Z}}$ is SE.

B2. The following event has positive probability of occurring:

$$\eta_t^2 \leq \inf_{x\in I} \frac{kx\eta_{t-1}^2}{\omega + \tilde{u}(x, \eta_{t-1}^2; \theta)} \quad \text{and} \quad \eta_{t-1}^2 \leq \inf_{x\in I} \frac{kx\eta_{t-2}^2}{\omega + \tilde{u}(x, \eta_{t-2}^2; \theta)}. \tag{2.7}$$

Assumption B is very general, but quite complex and thus hard to interpret. It is a restriction on the joint probability law of $(\eta_t, \eta_{t-1}, \eta_{t-2})$ that confines η_t and η_{t-1} with positive probability to an area described by the functions in (2.7). This area can be abstract and depends on the parameters k and θ. In what follows we derive a condition that is easier to verify than Assumption B2 by only focussing on this area close to the origin. Note that if η_t and η_{t-1} given η_{t-2} can be arbitrarily small with positive probability, then Assumption B2 is satisfied if the infima are nonzero. To that end we define the function

$$g(\eta;\theta) := \sup_{x\in I} \frac{\tilde{u}(x,\eta;\theta)}{x}.$$

Assumption C.

C1. For all $\eta \in \mathbb{R}$ and $\theta \in \Theta$ we have $g(\eta;\theta) < \infty$.

C2. The sequence $(\eta_t)_{t\in\mathbb{Z}}$ is SE.

C3. There exist a $N \in \mathbb{N}$ such that $\mathbb{P}(|\eta_t| < 1/n; |\eta_{t-1}| < 1/m \mid \eta_{t-2}) > 0$ almost surely for all $n, m \geq N$. Also the probability that $\eta_t = 0$ is zero.

Assumption C1 is an assumption on the updating function u of model (2.4). The condition is of a similar nature as those found in theory on geometric ergodicity of nonlinear time series, see Cline and Pu (1999). It implies that the function $\tilde{u}$ as a function of x is bounded on any closed interval, and asymptotically as $x \to \infty$ is bounded by a linear function. These two facts ensure that the infima in (2.7) are nonzero.

The other conditions are purely on the distribution of $(\eta_t)_{t\in\mathbb{Z}}$. Assumption C3 entails that η_t and η_{t-1} have positive probability of being arbitrarily small, independent of the value of η_{t-2}. An example on how Assumption C3 can be derived is if $(\eta_t)_{t\in\mathbb{Z}}$ is obtained as a SE solution from another model. For example, suppose that $(\eta_t)_{t\in\mathbb{Z}}$ is given by a SE solution to an autoregressive process of order one

$$\eta_{t+1} = \beta\eta_t + \zeta_t.$$

Then a sufficient condition would be that $(\zeta_t)_{t\in\mathbb{Z}}$ is iid, that ζ_t is absolutely continuous with respect to the Lebesque measure on $\mathbb{R}$ and that ζ_t has a strictly positive probability density function. Note that these conditions imply that any set in $B \in \mathcal{B}(\mathbb{R} \times \mathbb{R} \times \mathbb{R})$ has $\mathbb{P}((\eta_t, \eta_{t-1}, \eta_{t-2}) \in B) > 0$, so in particular Assumption C3 is implied.

If we can assume that the sequence $(\eta_t)_{t \in \mathbb{Z}}$ is independent, then Assumption C simplifies as follows:

Assumption D.

D1. For all $\eta \in \mathbb{R}$ and $\theta \in \Theta$ we have $g(\eta; \theta) < \infty$.

D2. The sequence $(\eta_t)_{t \in \mathbb{Z}}$ is iid.

D3. There exist a $N \in \mathbb{N}$ such that $\mathbb{P}(|\eta_t| < 1/n) > 0$ for all $n \geq N$. Also the probability that $\eta_t = 0$ is zero.

Assumption D3 implies Assumption C3 if $(\eta_t)_{t \in \mathbb{Z}}$ is iid and describes that η_t being arbitrarily small has positive probability. This, for example, is implied if η_t is absolutely continuous with respect to the Lebesque measure on $\mathbb{R}$ and the probability density function of η_t is strictly positive on an open interval around zero. Common distributions such as the normal and student-t distribution satisfy this condition.

Lemma 2.3.1. *Assumption C implies Assumption B.*

PROOF. We need to check whether Assumption B2 is satisfied. Assumption C1 ensures that the random variable

$$\inf_{x \in I} \frac{kx\eta_{t-1}^2}{\omega + \tilde{u}(x, \eta_{t-1}^2; \theta)}$$

is equal to zero if and only if $\eta_{t-1} = 0$, since

$$\inf_{x \in I} \frac{kx\eta_{t-1}^2}{\omega + \tilde{u}(x, \eta_{t-1}^2; \theta)} \geq \inf_{x \in I} \frac{kx\eta_{t-1}^2}{\omega + g(\eta_{t-1}; \theta)x} = \frac{k\eta_{t-1}^2}{1 + g(\eta_{t-1}; \theta)}.$$

Assumption C3 therefore implies that $k\eta_{t-1}^2/(1 + g(\eta_{t-1}; \theta))$ is nonzero with probability one. Therefore, the probability that

$$\eta_t^2 \leq \frac{k\eta_{t-1}^2}{1 + g(\eta_{t-1}; \theta)} \quad \text{and} \quad \eta_{t-1}^2 \leq \frac{k\eta_{t-2}^2}{1 + g(\eta_{t-2}; \theta)}$$

is greater than zero. This follows, since the infima are nonzero, due to the fact that η_t and η_{t-1} can be arbitrarily small with positive probability so in particular, they have positive probability to be smaller than these upper bounds. ■

Theorem 2.3.2. *If Assumption B holds, then there exists a solution $((\epsilon_t, \sigma_t^2))_{t\in\mathbb{Z}}$ to* (2.4) *given by*

$$\begin{aligned} \sigma_{t+1}^2 &= \lim_{m\to\infty} \phi_t^{(m)}(x,y), \\ \epsilon_{t+1} &= \sqrt{\lim_{m\to\infty} \phi_t^{(m)}(x,y)}\eta_{t+1}. \end{aligned} \tag{2.8}$$

This solution is stationary ergodic, unique and any partial solution converges to it at any rate. Moreover, if additionally $(\eta_t)_{t\in\mathbb{Z}}$ is φ-mixing with geometric rate, then $((\epsilon_t, \sigma_t^2))_{t\in\mathbb{Z}}$ is φ-mixing with geometric rate.

PROOF. We will start by verifying that assumptions A are all satisfied, so that Theorem 2.2.1 implies that

$$\left(\lim_{m\to\infty} \psi_t^{(m)}(x,y)\right)_{t\in\mathbb{Z}}$$

is a SE and unique solution to (2.6) such that all partial solutions converge to it. Assumption A1 is satisfied by Borel-measurability of u. Assumption A2 requires the sequence $((\eta_t, \eta_{t-1}))_{t\in\mathbb{Z}}$ to be SE, which is implied by B1 and Lemma 2.2.2. Finally, we will show that (2.7) implies that $\psi_t^{(3)}(x,y) = (\omega,\omega)$ for all $(x,y) \in S$ and therefore implies Assumption A3. Note that

$$\phi_t^{(2)}(x,y) = \phi_t(\phi_{t-1}(x,y), x) = \omega + \tilde{u}(\phi_{t-1}(x,y), \eta_t; \theta)\mathbb{1}\left\{\phi_{t-1}(x,y)\eta_t^2 > kx\eta_{t-1}^2\right\},$$

so that $\phi_t^{(2)}(x,y) = \omega$ for all $(x,y) \in S$ iff $\eta_t^2 \le \frac{kx\eta_{t-1}^2}{\phi_{t-1}(x,y)}$ for all $(x,y) \in S$, which is implied by

$$\eta_t^2 \le \inf_{x\in I} \frac{kx\eta_{t-1}^2}{\omega + \tilde{u}(x, \eta_{t-1}^2; \theta)}.$$

The first part of the proof is concluded by noting that

$$\psi_t^{(3)}(x,y) = (\phi_t^2(\phi_{t-2}(x,y),x), \phi_{t-1}^2(x,y)) = (\phi_t^2(\tilde{x},\tilde{y}), \phi_{t-1}^2(x,y)).$$

Next, a unique and SE solution to (2.6) to which all partial solutions converge to implies the existence of a solution to (2.5) with the same properties, by projecting on the first coordinate. The found solution is given by

$$\lim_{m\to\infty} \phi_t^{(m)}(x,y),$$

which is a measurable function of $(\eta_{t-1}, \eta_{t-2}, \ldots)$. Therefore $\epsilon_t = \sigma_t \eta_t$ is a measurable function of $(\eta_t, \eta_{t-1}, \ldots)$ and thus (2.8) is a SE solution to (2.4) by Lemma 2.2.2. Uniqueness and convergence of partial solutions transfer directly from those properties for (2.5).

Finally, suppose $(\eta_t)_{t\in\mathbb{Z}}$ is φ-mixing with geometric rate. Then $((\eta_t, \eta_{t-1}))_{t\in\mathbb{Z}}$ is φ-mixing with geometric rate by Lemma 2.2.5 and thus

$$\left(\lim_{m\to\infty} \phi_t^{(m)}(x,y), \eta_{t+1}\right)_{t\in\mathbb{Z}}$$

is φ-mixing with geometric rate by applying Theorem 2.2.4 and Lemma 2.2.5 again. Applying Lemma 2.2.5 once more shows that (2.8) is φ-mixing with geometric rate. ■

2.3.1 Examples

This section discusses a couple of specifications of the updating function u in model (2.4). We assume that the sequence $(\eta_t)_{t\in\mathbb{Z}}$ is φ-mixing at a geometric rate and satisfies the distributional conditions of either Assumption C or Assumption D. We then display how quickly our theory can be applied by checking whether Assumption C1/D1 holds for these examples.

Example 1 (Saïdi and Zakoian (2006))**.** First, we consider model (2.1). We repeat it here for readability.

$$\begin{aligned}\epsilon_t &= \sigma_t \eta_t, \\ \sigma_t^2 &= \omega + \alpha \epsilon_{t-1}^2 \mathbb{1}\left\{\epsilon_{t-1}^2 > k\epsilon_{t-2}^2\right\},\end{aligned}$$

where α is nonnegative. We have $u(\epsilon_{t-1}, \sigma_{t-1}^2; \alpha) = \alpha\epsilon_{t-1}^2$, which is a measurable and nonnegative function. Moreover, the function $g(\eta_t; \alpha) = \alpha < \infty$, so Assumption C1 respective D1 is immediately satisfied. Therefore there exists a strictly stationary and φ-mixing at geometric rate solution to which all partial solutions converge almost surely. Saïdi and Zakoian (2006) assume that $(\eta_t)_{t\in\mathbb{Z}}$ is iid. They then add the assumptions that η_t is absolutely continuous with respect to the Lebesque measure on $\mathbb{R}$ and η_t has a strictly positive probability density function. Note that this assumption is stronger than our Assumption D3. Finally, Saïdi and Zakoian (2006) assume that $\mathbb{E}\eta_t = 0$ and $\mathbb{E}\eta_t^2 = 1$, while we don't have any moment conditions at all.

Example 2 (Asymmetric news impact curve)**.** Second, we consider a model that allows the update function to be asymmetric in ϵ_{t-1} rather than using the quadratic update ϵ_{t-1}^2 considered above. In particular, we follow Engle and Ng (1993) in using the asymmetric news impact curve $u(\epsilon_{t-1}, \sigma_{t-1}^2) = \alpha(\epsilon_{t-1} + \delta\sigma_{t-1})^2$ and obtain the following model

$$\begin{aligned}\epsilon_t &= \sigma_t \eta_t, \\ \sigma_t^2 &= \omega + \alpha(\epsilon_{t-1} + \delta\sigma_{t-1})^2 \mathbb{1}\left\{\epsilon_{t-1}^2 > k\epsilon_{t-2}^2\right\},\end{aligned}$$

where α is nonnegative and $\delta \in \mathbb{R}$. Notice how for $\delta < 0$, negative returns ϵ_t have greater impact on future volatility σ_{t+1}^2 than positive returns of the same magnitude, thus capturing the empirical regularity known as the *leverage effect*. In this example we have $\tilde{u}(x, \eta_t; \alpha) = \alpha x(\eta_t + \delta)^2$ and thus $g(\eta_t; \alpha) = \alpha(\eta_t + \delta)^2 < \infty$. Therefore, Assumption C1/D1 is satisfied again and thus there exists a strictly stationary and φ-mixing at geometric rate solution to which all partial solutions converge almost surely.

Example 3 (Robust volatility update)**.** Finally, we consider a robust nonlinear ARCH model by adopting an update function that is bounded in ϵ_{t-1} rather than quadratic. In particular, we study a model which embodies the news impact curve of the student-t score volatility model introduced in Creal et al. (2011, 2013) and the beta-t EGARCH model proposed by Harvey (2013),

$$\begin{aligned}\epsilon_t &= \sigma_t \eta_t, \quad \eta_t \sim t(\lambda) \\ \sigma_t^2 &= \omega + \alpha \frac{\epsilon_{t-1}^2}{1+\lambda^{-1}\epsilon_{t-1}^2} \mathbb{1}\left\{\epsilon_{t-1}^2 > k\epsilon_{t-2}^2\right\},\end{aligned}$$

where α and λ are nonnegative. Notice that the innovations η_t are allowed to be fat tailed. In particular, they belong to the family of student's-t distributed random variables with λ degrees of freedom. The updating function of this model becomes more robust (with a lower upper bound) as $\lambda \to 0$ so that the innovations η_t become fatter tailed and outliers become more frequent. In contrast, as we approach the Gaussian case by letting $\lambda \to \infty$, then the updating function reverts back to that of the nonlinear ARCH model considered in Saïdi and Zakoian (2006). We now have $\tilde{u}(x, \eta_t; \alpha, \lambda) = \alpha \frac{x\eta_t^2}{1+x\eta_t^2/\lambda} \leq \alpha\lambda$, thus $g(\eta_t; \alpha, \lambda) \leq \alpha\lambda/\omega < \infty$ and Assumption C1/D1 is satisfied again. Hence, there exists a strictly stationary and φ-mixing at geometric rate solution to which all partial solutions converge almost surely.

2.3.2 Moments

Moment conditions for model (2.4) can be obtained by showing that the moments of the backward iterates have a converging subsequence. To state our result we define

$$h(\eta; \theta) = \limsup_{x\to\infty} \frac{\tilde{u}(x, \eta; \theta)}{x}.$$

Theorem 2.3.3. *Let Assumption D hold. Let $p \geq 1$ and $\tilde{\Theta} \subseteq \Theta$ be such that $\mathbb{E}|\eta_t|^{2p} < \infty$ and $\mathbb{E}\, g(\eta_t; \theta)^p < \infty$ and*

$$\mathbb{E}\left(h(\eta_t; \theta)h(\eta_{t-1}; \theta)\mathbb{1}\left\{\eta_t^2 > \frac{k\eta_{t-1}^2}{h(\eta_{t-1}; \theta)}\right\}\right)^p < 1 \tag{2.9}$$

for all $\theta \in \tilde{\Theta}$. Then the unique solution to (2.8) has finite absolute $2p$'th moment, that is $\mathbb{E}|\epsilon_t|^{2p} < \infty$ and $\mathbb{E}\sigma_t^{2p} < \infty$.

Theorem 2.3.3 is a generalisation of Theorem 3.3 in Saïdi and Zakoian (2006), their assumption to ensure moments in model (2.1) follows as a specific case from our result.

The expectation in condition (2.9) can be hard to calculate, because of the indicator function.

Corollary 2.3.4. *Condition* (2.9) *is implied by*

$$\mathbb{E}h(\eta_t;\theta)^p < 1. \tag{2.10}$$

PROOF. This follows directly from Assumption D2 and the fact that the indicator function is bounded by one. ■

Condition (2.10) is much easier to calculate, but sacrifices flexibility by ignoring the indicator function. Saïdi and Zakoian (2006) show that (2.9) delivers more flexible bounds for model (2.1) than (2.10) when $\eta_t \sim \mathrm{N}(0,1)$. We will discuss the examples of Section 2.3.1 to illustrate how both conditions can be useful.

PROOF OF THEOREM 2.3.3. By Assumption D2 we have $\mathbb{E}|\epsilon_t|^{2p} = \mathbb{E}|\eta_t|^{2p}\mathbb{E}\sigma_t^{2p}$, so we only have to show $\mathbb{E}\sigma_t^{2p} < \infty$. We know by theorem 2.3.2 that

$$\sigma_t^2 = \lim_{m\to\infty} \phi_t^{(m)}(x,y),$$

so by continuity of the norm and Fatou's lemma we have $\mathbb{E}\sigma_t^{2p} < \infty$ if

$$\liminf_{m\to\infty} \mathbb{E}\left|\phi_t^{(m)}(x,y)\right|^p < \infty. \tag{2.11}$$

We will prove inequality (2.11). To ease notation we will write $\phi_t^m = \phi_t^{(m)}(x,y)$ and suppress the dependence of the functions g and h on θ. We have

$$\begin{aligned}
\phi_t^m &= \omega + \tilde{u}\left(\phi_{t-1}^{m-1}, \eta_{t-1}\right) \mathbb{1}\left\{\phi_{t-1}^{m-1}\eta_{t-1}^2 > k\phi_{t-2}^{m-2}\eta_{t-2}^2\right\} \\
&\leq \omega + g(\eta_{t-1})\phi_{t-1}^{m-1} \\
&\leq \omega + g(\eta_{t-1})(\omega + g(\eta_{t-2})\phi_{t-2}^{m-2})
\end{aligned}$$

Let $n \in \mathbb{N}$ be any integer. We separate the problem into three scenarios. Suppose $\phi_{t-1}^{m-1} \leq$

n, then ϕ_t^m is bounded by

$$\omega + g(\eta_{t-1})n. \tag{2.12}$$

If $\phi_{t-2}^{m-2} \leq n$, then ϕ_t^m is bounded by

$$\omega + g(\eta_{t-1})(\omega + g(\eta_{t-2})n). \tag{2.13}$$

Finally, suppose $\phi_{t-1}^{m-1}, \phi_{t-2}^{m-2} \geq n$. Define

$$h_n(\eta) = \sup_{x \geq n} \frac{\tilde{u}(x, \eta; \theta)}{x}.$$

Then, for $n \geq \omega$, we have $h_n(\eta) \leq g(\eta)$ and thus

$$\begin{aligned} \phi_t^m &\leq \omega + h_n(\eta_{t-1})\phi_{t-1}^{m-1}\mathbb{1}\left\{\eta_{t-1}^2 > \frac{k\phi_{t-2}^{m-2}\eta_{t-2}^2}{\omega + \tilde{u}\left(\phi_{t-2}^{m-2}, \eta_{t-2}\right)}\right\} \\ &\leq (1 + g(\eta_{t-1}))\omega + h_n(\eta_{t-1})h_n(\eta_{t-2})\phi_{t-2}^{m-2}\mathbb{1}\left\{\eta_{t-1}^2 > \frac{k\eta_{t-2}^2}{\omega/n + h_n(\eta_{t-2})}\right\}. \end{aligned} \tag{2.14}$$

It follows that ϕ_t^m is bounded by the sum of (2.12)-(2.14) and therefore by independence of ϕ_{t-2}^{m-2} with η_{t-1} and η_{t-2} we get by Minkowski's inequality that

$$[\mathbb{E}\left(\phi_t^m\right)^p]^{\frac{1}{p}} \leq C(n) + [\mathbb{E}f_n(\eta_{t-1}, \eta_{t-2})^p]^{\frac{1}{p}} \left[\mathbb{E}\left(\phi_{t-2}^{m-2}\right)^p\right]^{\frac{1}{p}},$$

where $C(n)$ is a finite constant depending on n and

$$f_n(\eta_{t-1}, \eta_{t-2}) = h_n(\eta_{t-1})h_n(\eta_{t-2})\mathbb{1}\left\{\eta_{t-1}^2 > \frac{k\eta_{t-2}^2}{\omega/n + h_n(\eta_{t-2})}\right\}.$$

A sufficient condition for (2.11) is to find an appropriate $n \in \mathbb{N}$ such that the expectation $\mathbb{E}f_n(\eta_{t-1}, \eta_{t-2})^p < 1$. This happens for any choice of n that is large enough, as implied by (2.9) and the dominated convergence theorem, because $f_n(\eta_{t-1}, \eta_{t-2})$ is bounded by $g(\eta_{t-1})g(\eta_{t-2})$ for large enough n and as $n \to \infty$ it converges pointwise to

$$h(\eta_{t-1})h(\eta_{t-2})\mathbb{1}\left\{\eta_{t-1}^2 > \frac{k\eta_{t-2}^2}{h(\eta_{t-2})}\right\}.$$

■

Example 1 (Saïdi and Zakoian (2006) continued)**.** Using Theorem 2.3.3 we can follow the approach of Saïdi and Zakoian (2006) and find the same conditions for model (2.1) that ensure $\mathbb{E}|\epsilon_t|^{2p} < \infty$ and $\mathbb{E}\sigma_t^{2p} < \infty$. We need $\mu_{2p} := \mathbb{E}|\eta_t|^{2p} < \infty$ and note that it implies $\mathbb{E}g(\eta_t;\alpha)^p = \alpha^p\mu_{2p} < \infty$. In this example condition (2.9) boils down to

$$\mathbb{E}\left(\alpha^2\eta_t^2\eta_{t-1}^2\mathbb{1}\left\{\eta_t^2 > \frac{k}{\alpha}\right\}\right)^p < 1. \tag{2.15}$$

Using Hölder's and Markov's inequalities we get for any $m \in \mathbb{N}$ that the expectation in (2.15) is bounded by

$$\begin{aligned}\alpha^{2p}\mu_{2p}\mathbb{E}\left(\eta_t^2\left\{\eta_t^2 > \frac{k}{\alpha}\right\}\right)^p &\leq \alpha^{2p}\mu_{2p}\mu_{2pm}^{1/m}\,\mathbb{P}\left(\eta_t^{2m} > \left(\frac{k}{\alpha}\right)^m\right)^{\frac{m-1}{m}}\\ &\leq \alpha^{2p}\mu_{2p}\mu_{2pm}^{1/m}\mu_{2m}^{(m-1)/m}\left(\frac{\alpha}{k}\right)^{m-1}\end{aligned}$$

Therefore a sufficient condition for (2.9) is

$$\alpha < \max_{m\in\mathbb{N}}\left(\frac{k^{m-1}}{\mu_{2p}\mu_{2pm}^{1/m}\mu_{2m}^{(m-1)/m}}\right)^{1/(2p+m-1)}.$$

Example 2 (Asymmetric news impact curve continued)**.** The model with leverage effects requires again $\mu_{2p} < \infty$, which implies $\mathbb{E}g(\eta_t;\alpha)^p = \alpha^p\mathbb{E}(\eta_t+\delta)^{2p} \leq 2^{2p-1}\alpha^p(\mu_{2p} + |\delta|^{2p}) < \infty$. This model provides an example where the expectation in (2.9) is hard to calculate. The condition here leads to

$$\mathbb{E}\left(\alpha^2(\eta_t+\delta)^2(\eta_{t-1}+\delta)^2\mathbb{1}\left\{\eta_t^2 > \frac{k\eta_{t-1}^2}{\alpha(\eta_{t-1}+\delta)^2}\right\}\right)^p < 1,$$

but we cannot easily use the Markov inequality to bound the indicator function, since this would lead to moments of the reciprocal of η_t. Instead we use (2.10) and get the sufficient condition $\alpha < [\mathbb{E}(\eta_t + \delta)^{2p}]^{-1/p}$ to obtain $\mathbb{E}|\epsilon_t|^{2p} < \infty$ and $\mathbb{E}\sigma_t^{2p} < \infty$.

Example 3 (Robust volatility update continued)**.** The robust model has a bounded updating function for the volatility, so therefore we immediately know that $\mu_{2p} < \infty$ is the only condition we need $\mathbb{E}|\epsilon_t|^{2p} < \infty$ and $\mathbb{E}\sigma_t^{2p} < \infty$. This result also follows from Theorem 2.3.3, since $g(\eta; \alpha, \lambda) \leq \alpha\lambda$ and $h(\eta; \alpha, \lambda) = 0$ for all $\eta \in \mathbb{R}$.

2.4 Conclusion

This chapter has introduced a new set of conditions that ensure the existence of a unique stationary, ergodic and φ-mixing solution for time series models. Moreover, sample paths are guaranteed to converge to this solution over time. The assumptions are different from existing conditions as they do not impose Lipschitz, bounded growth or drift restrictions. Instead we require that the time series contains resetting dynamics, where a reset implies that the model has a positive probability to update to a value that does not depend on the past. These dynamics are present in time series with sudden changes, such as stock prices with financial bubbles. We have demonstrated the value of our results and illustrated how to apply them by examining a generalisation of the nonlinear ARCH model studied in Saïdi and Zakoian (2006).